# Multiplicativity of $l_p$ Norms for Matrices. II

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ABSTRACT

For $1 \leqslant p \leqslant \infty$, let

$$|A|_p = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^p \right)^{1/p},$$

be the $l_p$ norm of an $m \times n$ complex matrix $A = (\alpha_{ij}) \in \mathbf{C}_{m \times n}$. The main purpose of this paper is to find, for any $p, q \geqslant 1$, the best (smallest) possible constants $\tau(m, k, n, p, q)$ and $\sigma(m, k, n, p, q)$ for which inequalities of the form

$$|AB|_p \leqslant \tau(m,k,n,p,q)|A|_p|B|_q, \qquad |AB|_p \leqslant \sigma(m,k,n,p,q)|A|_q|B|_p$$

hold for all $A \in \mathbf{C}_{m \times k}$, $B \in \mathbf{C}_{k \times n}$. This leads to upper bounds for inner products on $\mathbf{C}^k$ and for ordinary $l_p$ operator norms on $\mathbf{C}_{m \times n}$.

*Research sponsored in part by Air Force Office of Scientific Research, Air Force System Command, Grant AFOSR-83-0150.




52 Vanderbilt Ave., New York, NY 10017 0024-3795/84/$3.00

## 1. INTRODUCTION AND STATEMENT OF RESULTS

The $l_p$ norm, $1 \leqslant p \leqslant \infty$, of an $m \times n$ complex matrix $A = (\alpha_{ij}) \in \mathbf{C}_{m \times n}$ is defined by

$$|A|_p = \left( \sum_{i=1}^{m} \sum_{j=1}^{n} |\alpha_{ij}|^p \right)^{1/p},$$

where for $p = \infty$ (a case which need not be treated separately) we have

$$|A|_\infty = \max_{i,j} |\alpha_{ij}|.$$

Studying these $l_p$ norms, Ostrowski [4] obtained the following results:

THEOREM 1.1 [4, Theorem 7]. *If $1 \leqslant p \leqslant 2$ and if $A, B$ are rectangular matrices such that $AB$ exists, then*

$$|AB|_p \leqslant |A|_p |B|_p.$$

THEOREM 1.2 [4, Theorem 8]. *If $1 \leqslant q \leqslant 2 \leqslant p$, $1/p + 1/q = 1$, and if $A, B$ are rectangular matrices such that $AB$ exists, then*

$$|AB|_p \leqslant |A|_p |B|_q,$$

$$|AB|_p \leqslant |A|_q |B|_p.$$

While Ostrowski proved that for $1 \leqslant p < 2 < q$ the inequalities in Theorems 1.1, 1.2 may fail, these theorems were extended in [2] as follows:

THEOREM 1.3 [2, Theorem 1.3]. *If $p \geqslant 2$ and if $A \in \mathbf{C}_{m \times k}$, $B \in \mathbf{C}_{k \times n}$, then*

$$|AB|_p \leqslant k^{1-2/p} |A|_p |B|_p.$$

THEOREM 1.4 [2, Theorem 1.4]. *If* $1 \leqslant p \leqslant 2 \leqslant q$, $1/p + 1/q = 1$, *and* $A \in \mathbf{C}_{m\times k}$, $B \in \mathbf{C}_{k\times n}$, *then*

$$|AB|_p \leqslant n^{1-2/q}|A|_p|B|_q,$$

$$|AB|_p \leqslant m^{1-2/q}|A|_q|B|_p.$$

In this paper we generalize Theorems 1.1–1.4 as follows:

THEOREM 1.1′. *Let* $A \in \mathbf{C}_{m\times k}$ *and* $B \in \mathbf{C}_{k\times n}$. *If* $1 \leqslant q \leqslant p \leqslant 2$, *then*

$$|AB|_p \leqslant |A|_p|B|_q, \tag{1.1}$$

$$|AB|_p \leqslant |A|_q|B|_p; \tag{1.2}$$

*and if* $1 \leqslant p \leqslant q \leqslant 2$, *then*

$$|AB|_p \leqslant n^{1/p-1/q}|A|_p|B|_q, \tag{1.3}$$

$$|AB|_p \leqslant m^{1/p-1/q}|A|_q|B|_p. \tag{1.4}$$

THEOREM 1.2′. *Let* $A \in \mathbf{C}_{m\times k}$, $B \in \mathbf{C}_{k\times n}$, *and* $1 \leqslant q \leqslant 2 \leqslant p$. *If* $1/p + 1/q \geqslant 1$, *then*

$$|AB|_p \leqslant |A|_p|B|_q, \tag{1.5}$$

$$|AB|_p \leqslant |A|_q|B|_p; \tag{1.6}$$

*and if* $1/p + 1/q \leqslant 1$, *then*

$$|AB|_p \leqslant k^{1-1/p-1/q}|A|_p|B|_q, \tag{1.7}$$

$$|AB|_p \leqslant k^{1-1/p-1/q}|A|_q|B|_p. \tag{1.8}$$

THEOREM 1.3′. *Let* $A \in \mathbf{C}_{m\times k}$ *and* $B \in \mathbf{C}_{k\times n}$. *If* $p \geqslant q \geqslant 2$, *then*

$$|AB|_p \leqslant k^{1-1/p-1/q}|A|_p|B|_q, \tag{1.9}$$

$$|AB|_p \leqslant k^{1-1/p-1/q}|A|_q|B|_p; \tag{1.10}$$

*and if* $q \geqslant p \geqslant 2$, *then*

$$|AB|_p \leqslant k^{1-1/p-1/q} n^{1/p-1/q} |A|_p |B|_q, \tag{1.11}$$

$$|AB|_p \leqslant k^{1-1/p-1/q} m^{1/p-1/q} |A|_q |B|_p. \tag{1.12}$$

THEOREM 1.4′. *Let* $A \in \mathbf{C}_{m \times k}$, $B \in \mathbf{C}_{k \times n}$, *and* $1 \leqslant p \leqslant 2 \leqslant q$. *If* $1/p + 1/q \geqslant 1$, *then*

$$|AB|_p \leqslant n^{1/p-1/q} |A|_p |B|_q, \tag{1.13}$$

$$|AB|_p \leqslant m^{1/p-1/q} |A|_q |B|_p; \tag{1.14}$$

*and if* $1/p + 1/q \leqslant 1$, *then*

$$|AB|_p \leqslant k^{1-1/p-1/q} n^{1/p-1/q} |A|_p |B|_q, \tag{1.15}$$

$$|AB|_p \leqslant k^{1-1/p-1/q} m^{1/p-1/q} |A|_q |B|_p. \tag{1.16}$$

Theorems 1.1′–1.4′—proved in Section 2—obviously cover all possible relations between $p$ and $q$, for $p, q \geqslant 1$. Moreover, for $p = q$, Theorems 1.1′ and 1.3′ reduce to Theorems 1.1 and 1.3, respectively; and taking $1/p + 1/q = 1$, Theorems 1.2′, 1.4′ yield Theorems 1.2 and 1.4; hence, Theorems 1.1′–1.4′ indeed generalize Theorems 1.1–1.4.

If $A = a = (\alpha_1, \dots, \alpha_k)$ is a row vector and $B = b^* = (\beta_1, \dots, \beta_k)^*$ is a column vector (* denoting the adjoint), then $AB = ab^*$ is the standard inner product $(a, b)$ on $\mathbf{C}^k$. Applying Theorems 1.1′–1.4′ to this case and noting that $|b^*|_p = |b|_p$, we immediately obtain:

COROLLARY 1.1. *Let* $a, b \in \mathbf{C}^k$ *be k-vectors. Then for* $1 \leqslant p, q \leqslant 2$,

$$|(a, b)| \leqslant |a|_p |b|_q; \tag{1.17}$$

*for* $p, q \geqslant 2$,

$$|(a, b)| \leqslant k^{1-1/p-1/q} |a|_p |b|_q; \tag{1.18}$$

*and for* $1 \leqslant p \leqslant 2 \leqslant q$ *or* $1 \leqslant q \leqslant 2 \leqslant p$,

$$|(a, b)| \leqslant |a|_p |b|_q, \qquad \frac{1}{p} + \frac{1}{q} \geqslant 1, \tag{1.19}$$

$$|(a, b)| \leqslant k^{1-1/p-1/q} |a|_p |b|_q, \qquad \frac{1}{p} + \frac{1}{q} \leqslant 1. \tag{1.20}$$

Setting $p = q$ in (1.17) and (1.18), we obtain the two inequalities,

$$|(a,b)| \leqslant |a|_p |b|_p, \qquad 1 \leqslant p \leqslant 2,$$

$$|(a,b)| \leqslant k^{1-2/p} |a|_p |b|_p, \qquad p \geqslant 2,$$

which were observed already in [2]. If $p$ and $q$ are conjugate, i.e., $1/p + 1/q = 1$, then (1.19), (1.20) yield Hölder's inequality:

$$|(a,b)| \leqslant |a|_p |b|_q, \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad p \geqslant 1.$$

In addition to the above $l_p$ norms, let us also consider the usual $l_p$ *operator norm*, $1 \leqslant p \leqslant \infty$, of a matrix $A = (\alpha_{ij}) \in \mathbf{C}_{m \times n}$:

$$\|A\|_p = \max\{ |Ax|_p : x \in \mathbf{C}^n, \ |x|_p = 1 \}.$$

Using Theorems 1.1′–1.4′, we shall easily prove in Section 2:

THEOREM 1.5. *Let $A \in \mathbf{C}_{m \times n}$. Then for $1 \leqslant q \leqslant p \leqslant 2$,*

$$\|A\|_p \leqslant |A|_q; \tag{1.21}$$

*for $1 \leqslant p \leqslant q \leqslant 2$,*

$$\|A\|_p \leqslant m^{1/p - 1/q} |A|_q; \tag{1.22}$$

*for $1 \leqslant q \leqslant 2 \leqslant p$, $1/p + 1/q \geqslant 1$,*

$$\|A\|_p \leqslant |A|_q; \tag{1.23}$$

*for $1 \leqslant q \leqslant 2 \leqslant p$, $1/p + 1/q \leqslant 1$,*

$$\|A\|_p \leqslant n^{1 - 1/p - 1/q} |A|_q; \tag{1.24}$$

*for $p \geqslant q \geqslant 2$,*

$$\|A\|_p \leqslant n^{1 - 1/p - 1/q} |A|_q; \tag{1.25}$$

*for* $q \geqslant p \geqslant 2$,

$$\|A\|_p \leqslant m^{1/p-1/q} n^{1-1/p-1/q} |A|_q; \tag{1.26}$$

*for* $1 \leqslant p \leqslant 2 \leqslant q$, $1/p + 1/q \geqslant 1$,

$$\|A\|_p \leqslant m^{1/p-1/q} |A|_q; \tag{1.27}$$

*and for* $1 \leqslant p \leqslant 2 \leqslant q$, $1/p + 1/q \leqslant 1$,

$$\|A\|_p \leqslant m^{1/p-1/q} n^{1-1/p-1/q} |A|_q. \tag{1.28}$$

Note that for the special cases $p = q$ and $1/p + 1/q = 1$, Theorem 1.5 reduces to Theorem 1.6 of [2]. In particular, for $p = q$ we obtain

$$\|A\|_p \leqslant \begin{cases} |A|_p, & 1 \leqslant p \leqslant 2, \\ n^{1-2/p} |A|_p, & p \geqslant 2. \end{cases}$$

As shown at the end of Section 2, choosing certain incidence [i.e., (0,1)] matrices, each of the inequalities in Theorems 1.1′–1.4′, 1.5, and 1.6, becomes an equality. Thus *none of the inequalities in this paper can be improved.* In particular, this implies that the results in Theorems 1.1′–1.4′ provide the best (smallest) possible constants, $\tau(m, k, n, p, q)$ and $\sigma(m, k, n, p, q)$, for which inequalities of the form

$$|AB|_p \leqslant \tau(m, k, n, p, q) |A|_p |B|_q, \qquad |AB|_p \leqslant \sigma(m, k, n, p, q) |A|_q |A|_p$$

hold for all $A \in \mathbf{C}_{m \times k}$, $B \in \mathbf{C}_{k \times n}$.

## 2. PROOFS

We start by quoting:

Lemma 2.1 ([3, Corollary I.4.5]; compare [1, Chapter 1, Section 16]). *If* $x \in \mathbf{C}^k$ *and if* $p \geqslant q \geqslant 1$, *then*

$$|x|_p \leqslant |x|_q \leqslant k^{1/q-1/p} |x|_p.$$

If we define for convenience

$$\mu_{pq}(k)=\begin{cases}1, & p\geqslant q\geqslant 1,\\ k^{1/p-1/q}, & q\geqslant p\geqslant 1,\end{cases}$$

then Lemma 2.1 can be restated as follows:

LEMMA 2.2. *For $x\in\mathbf{C}^k$ and $p,q\geqslant 1$,*

$$|x|_p\leqslant\mu_{pq}(k)|x|_q. \tag{2.1}$$

We next consider the mixed $l_{p,q}$ norm, $p,q\geqslant 1$, of a matrix $A=(\alpha_{ij})\in\mathbf{C}_{m\times n}$, [4],

$$|A|_{pq}=\left[\sum_{j=1}^{n}\left(\sum_{i=1}^{m}|\alpha_{ij}|^p\right)^{q/p}\right]^{1/q},$$

and prove:

LEMMA 2.3. *For $p,q\geqslant 1$ and $A\in\mathbf{C}_{m\times n}$,*

$$|A|_{pq}\leqslant\mu_{pq}(m)|A|_q, \tag{2.2}$$

$$|A|_{pq}\leqslant\mu_{qp}(n)|A|_p. \tag{2.3}$$

*Proof.* Denote the columns of $A$ by $a_1,\ldots,a_n$. Then (2.1) yields

$$\begin{aligned}|A|_{pq}&=\big|\big(|a_1|_p,\ldots,|a_n|_p\big)\big|_q\\ &\leqslant\big|\big(\mu_{pq}(m)|a_1|_q,\ldots,\mu_{pq}(m)|a_n|_q\big)\big|_q\\ &=\mu_{pq}(m)\big|\big(|a_1|_q,\ldots,|a_n|_q\big)\big|_q=\mu_{pq}(m)|A|_q,\end{aligned}$$

and (2.2) holds. By (2.1) again, we obtain

$$\begin{aligned}|A|_{pq}&=\big|\big(|a_1|_p,\ldots,|a_n|_p\big)\big|_q\\ &\leqslant\mu_{qp}(n)\big|\big(|a_1|_p,\ldots,|a_n|_p\big)\big|_p=\mu_{qp}(n)|A|_p,\end{aligned}$$

and the lemma follows. ■

LEMMA 2.4. *Let $A \in \mathbf{C}_{m\times k}$, $B \in \mathbf{C}_{k\times n}$. Let $p, q \geqslant 1$, and let $q'$ be the conjugate of $q$ so that $1/q + 1/q' = 1$. Then,*

$$|AB|_p \leqslant |A^T|_{q'p}|B|_{qp}, \tag{2.4}$$

$$|AB|_p \leqslant |A^T|_{qp}|B|_{q'p}, \tag{2.5}$$

*where $A^T$ denotes the transpose of $A$.*

*Proof.* Set $C = AB$, $C = (\gamma_{ij})$. By Hölder's inequality,

$$|\gamma_{ij}| = \left|\sum_{l=1}^{k} \alpha_{il}\beta_{lj}\right| \leqslant \left(\sum_{l=1}^{k} |\alpha_{il}|^{q'}\right)^{1/q'} \left(\sum_{l=1}^{k} |\beta_{lj}|^{q}\right)^{1/q}.$$

Hence,

$$\begin{aligned}
|AB|_p^p &= \sum_{i=1}^{m} \sum_{j=1}^{n} |\gamma_{ij}|^p \\
&\leqslant \sum_{i=1}^{m} \sum_{j=1}^{n} \left[\left(\sum_{l=1}^{k} |\alpha_{il}|^{q'}\right)^{1/q'} \left(\sum_{l=1}^{k} |\beta_{lj}|^{q}\right)^{1/q}\right]^p \\
&= \left[\sum_{i=1}^{m}\left(\sum_{l=1}^{k} |\alpha_{il}|^{q'}\right)^{p/q'}\right]\left[\sum_{j=1}^{n}\left(\sum_{l=1}^{k} |\beta_{lj}|^{q}\right)^{p/q}\right] \\
&= |A^T|_{q'p}^p |B|_{qp}^p,
\end{aligned}$$

and (2.4) is established. To obtain (2.5) we repeat the proof with $q$ and $q'$ exchanged. ∎

We are now ready for

*Proof of Theorems 1.1′–1.4′.* First use (2.4), (2.2), and (2.3) to obtain

$$\begin{aligned}
|AB|_p &\leqslant |A^T|_{q'p}|B|_{qp} \leqslant \mu_{q'p}(k)|A^T|_p \cdot \mu_{pq}(n)|B|_q \\
&= \mu_{q'p}(k)\mu_{pq}(n)|A|_p|B|_q.
\end{aligned} \tag{2.6}$$

Thus, if $1 \leqslant q \leqslant p \leqslant 2$, then $q' \geqslant p$, so (1.1) follows; and if $1 \leqslant p \leqslant q \leqslant 2$, then again $q' \geqslant p$, and (1.3) follows. If $1 \leqslant q \leqslant 2 \leqslant p$ and $1/p + 1/q \geqslant 1$, then $1/q' = 1 - 1/q \leqslant 1/p$, so $q' \geqslant p$, and (2.6) yields (1.5). Similarly, if $1 \leqslant q \leqslant 2 \leqslant p$ with $1/p + 1/q \leqslant 1$, then $q' \leqslant p$ and

$$\mu_{q'p}(k) = k^{1/q' - 1/p} = k^{1 - 1/p - 1/q}, \tag{2.7}$$

so (1.7) holds. If $p \geqslant q \geqslant 2$, then $p \geqslant q'$, so we have (2.7) again, and (2.6) gives (1.9); and if $q \geqslant p \geqslant 2$, then again $p \geqslant q'$, and we obtain (1.11). If $1 \leqslant p \leqslant 2 \leqslant q$ and $1/p + 1/q \geqslant 1$, then $q' \geqslant p$, and we get (1.13). Finally, if $1 \leqslant p \leqslant 2 \leqslant q$ with $1/p + 1/q \leqslant 1$, then by (2.6) and (2.7), we obtain (1.15).

Using (2.5), (2.3), and (2.2), we find instead of (2.6),

$$\begin{aligned}|AB|_p &\leqslant |A^T|_{qp}|B|_{q'p} \leqslant \mu_{pq}(m)|A^T|_q \cdot \mu_{q'p}(k)|B|_p \\ &= \mu_{pq}(m)\mu_{q'p}(k)|A|_q|B|_p.\end{aligned}$$

Hence, considering as before the eight cases, $1 \leqslant q \leqslant p \leqslant 2$, $1 \leqslant p \leqslant q \leqslant 2$, $1 \leqslant q \leqslant 2 \leqslant p$ with $1/p + 1/q \geqslant 1$, $1 \leqslant q \leqslant 2 \leqslant p$ with $1/p + 1/q \leqslant 1$, $p \geqslant q \geqslant 2$, $q \geqslant p \geqslant 2$, $1 \leqslant p \leqslant 2 \leqslant q$ with $1/p + 1/q \geqslant 1$, and $1 \leqslant p \leqslant 2 \leqslant q$ with $1/p + 1/q \leqslant 1$, we obtain (1.2), (1.4), (1.6), (1.8), (1.10), (1.12), (1.14), and (1.16), respectively. ■

*Proof of Theorem 1.5.* Suppose $1 \leqslant q \leqslant p \leqslant 2$. Then by (1.2), for any $n$-vector $x \in \mathbf{C}^n$(which we view as an $n \times 1$ matrix),

$$|Ax|_p \leqslant |A|_q|x|_p.$$

Thus,

$$\|A\|_p = \max_{|x|_p = 1} |Ax|_p \leqslant |A|_q,$$

so (1.21) follows. The other seven inequalities in Theorem 1.5 follow precisely along the same lines, where instead of (1.2) we use (1.4), (1.6), (1.8), (1.10), (1.12), (1.14), and (1.16), respectively. ■

We conclude the paper by showing that our inequalities in Theorems 1.1′–1.4′, 1.5, and 1.6, are sharp; that is, for certain matrices they become equalities. For this purpose we introduce four $m \times n$ incidence matrices,

$E(m,n)$, $R(m,n)$, $C(m,n)$, and $J(m,n)$, defined by

$$E(m,n)_{ij}=\delta_{i1}\delta_{j1},\quad R(m,n)_{ij}=\delta_{i1},\quad C(m,n)_{ij}=\delta_{j1},\quad J(m,n)_{ij}=1,$$
$$i=1,\dots,m,\quad j=1,\dots,n.$$

A direct calculation shows that for $A=E(m,k)$, $B=E(k,n)$, we obtain equality in (1.1), (1.2), (1.5), and (1.6); for $A=C(m,k)$, $B=R(k,n)$, equality holds in (1.3), (1.4), (1.13), and (1.14); for $A=R(m,k)$, $B=C(k,n)$, it holds in (1.7)–(1.10); for $A=R(m,k)$, $B=J(k,n)$, it holds in (1.11) and (1.15); and for $A=J(m,k)$, $B=C(k,n)$, it holds in (1.12) and (1.16).

The same types of incidence matrices (or vectors) provide equalities in (1.17)–(1.28).

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*Received 30 September 1983*